english

Combinatorial Counting - 3.3 Binomial Theorem

Binomial coefficient: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Multinomial coefficient: $\binom{n}{n_1\ n_2\ \cdots\ n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$, where $\sum_{i=1}^{k} n_i = n$.

1: Prove the following identity, which is a generalization of Pascal's formula for multinomial coefficients.

$$\binom{n}{n_1\ n_2\ \cdots\ n_k} = \binom{n-1}{n_1-1\ n_2\ \cdots\ n_k} + \binom{n-1}{n_1\ n_2-1\ \cdots\ n_k} + \cdots + \binom{n-1}{n_1\ n_2\ \cdots\ n_k-1}$$

Hint: Combinatorial verification might be more elegant.

Solution: Suppose you have n people and n_i pieces of candy i. How many ways can you give exactly one candy to everyone? Left hand side count them together. Right hand picks one person, assigns candy and then distributes the rest.

Theorem 3.3.3 *(Binomial Theorem - more general version)* For every integer $n > 0$ and all x and y

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}y^n \\ &= \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i\end{aligned}$$

2: Prove Binomial Theorem. (Hints: Investigate how the multiplication expands. Or think about an alphabet of $x+y$ letters and making n-letter words. Or induction.)

Solution: 1) $(x+y)^n = (x+y)\cdot(x+y)\cdot(x+y)\cdots(x+y)$.

From each $(x+y)$ pick x or y. Notice the sum of the exponents is always n. What is the coefficient?

2) Suppose alphabet of $x+y$ letters. How many words of length n can you make? $(x+y)^n$. What about number of words that contains exactly i letters from x? That is $\binom{n}{i}x^iy^{n-i}$.

Theorem 3.3.5. *(Multinomial theorem)* Let $n \in \mathbb{N}$. For all $x_1, \ldots, x_k$

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{n_1\ n_2\ \cdots\ n_k} x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k},$$

where $\sum$ is over all non-negative integral solutions of $n_1 + n_2 + \cdots + n_k = n$.

3: Prove the theorem by generalizing some of the proofs for binomial theorem.

Solution:

$$(x_1+x_2+\cdots+x_k)\cdot(x_1+x_2+\cdots+x_k)\cdots(x_1+x_2+\cdots+x_k) = \sum \alpha_{n_1,n_2,\ldots,n_k}x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$$

What is $\alpha_{n_1,n_2,\ldots,n_k}$? Product has n small sums. n_1 of them picks x_1. Then from the rest, n_2 picks x_2,.... This gives:

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots = \frac{n}{n_1!n_2!\cdots n}$$

Which is the multinomial coefficient.

4: What is the coefficient of x^5y^{13} in the expansion of $(3x-2y)^{18}$? What is the coefficient of x^8y^9?

Solution: The coefficient comes from

$$\binom{18}{5}(3x)^5(-2y)^{13} = -\binom{18}{5}3^5 2^{13}\cdot x^5y^{13}.$$

The coefficient of x^8y^9 is zero, since it does not appear in the expansion.

5: What is the coefficient of $x_1^2x_2x_3^2$ in the expansion of $(2x_1-4x_2-3x_3)^5$?

Solution:

$$\binom{5}{2\ 1\ 2}\cdot 2^2\cdot(-1)\cdot(-3)^2 = -1080$$

6: Expand $(x+1)^n$.

Solution:

$$(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

7: What happens with Binomial theorem if $x = 1$ and $y = 1$? Give a combinatorial interpretation of the resulting identity.

Solution:

$$(1+1)^n = \sum_{i=0}^{n}\binom{n}{i}$$

Number of all subsets is 2^n.

8: What happens with Binomial theorem if $x = 1$ and $y = -1$? Give a combinatorial interpretation of the resulting identity.

Solution:

$$(1-1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i$$

Number of all subsets of even size is equal to the number of all subsets of the odd size.

9: Prove that the sequence of numbers in each row of Pascal's triangle is a power of 11. I.e., "$1,2,1$" $\to 121 = 11^2$. For this you need to *carry over* numbers bigger than 9 to the left. So for example "1,5,10,10,5,1" gives "161051"$=11^5$.
Hint: How to build the result from numbers/digits in the triangle? By sum and multiplying by $10, 100, \ldots$*..?*

Solution: Use $x = 10$ in the binomial theorem for $(x+1)^n$. For $n = 5$ we have 1 5 10 10 5 1 which can be created as

$$1 + 5\cdot 10 + 10\cdot 10^2 + 10\cdot 10^3 + 5\cdot 10^4 + 1\cdot 10^5 =$$
$$\binom{5}{0} + \binom{5}{1}\cdot 10 + \binom{5}{2}\cdot 10^2 + \binom{5}{3}\cdot 10^3 + \binom{5}{4}\cdot 10^4 + \binom{5}{5}\cdot 10^5 =$$
$$\sum_{i=0}^{5}\binom{5}{i}10^i = (10+1)^5 = 11^5.$$

10: Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.
(Try to find also a combinatorial argument)

Solution: Think of committee and its chair. On the left, pick committee of k and then 1 chair. On the right pick chair and then the rest.

11: Show that

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

(Is it possible to do it from binomial theorem? Hint: derivative.)

Solution: a) We use that $k\binom{n}{k} = n\binom{n-1}{k-1}$ and start by summing all subsets of $n-1$ elements.

$$\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1} = 2^{n-1}$$
$$n\binom{n-1}{0} + n\binom{n-1}{1} + n\binom{n-1}{2} + \cdots + n\binom{n-1}{n-1} = n2^{n-1}$$
$$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$$

b) Use derivative on the binomial theorem

$$(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$
$$n(x+1)^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}x^{k-1}$$
$$n(1+1)^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}$$

12: Evaluate
$$\binom{n}{0} - 2\binom{n}{1} + 3\binom{n}{2} + \ldots + (-1)^n(n+1)\binom{n}{n}.$$

Solution: We need to derive a formula, and combine it with a previously stated one. Differentiate the binomial theorem

$$(x+1)^n = \sum_{0\le i\le n}\binom{n}{i}x^i$$

giving

$$n(x+1)^{n-1} = \sum_{1\le i\le n} i\binom{n}{i}x^{i-1}$$

Setting $x=-1$ and multiplying the LHS and RHS by -1 gives

$$0 = \sum_{1\le i\le n} i\binom{n}{i}(-1)^i = -\binom{n}{1} + 2\binom{n}{2} - 3\binom{n}{3} + \ldots + (-1)^n n\binom{n}{n}$$

Now add this to the identity obtained as $(x+1)^n$ for $x=-1$:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n\binom{n}{n} = 0$$

and get 0.

13: By integrating the binomial expansion, prove that, for any integer n,
$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$

Solution: By the binomial theorem we have

$$(x+1)^n = \sum_{j=0}^{n}\binom{n}{j}x^j$$

Integrating gives

$$\frac{(x+1)^{n+1}}{n+1} = \sum_{j=0}^{n}\binom{n}{j}\frac{x^{j+1}}{j+1} + C$$

where C is constant of integration. The constant C can be computed by setting $x=0$ and it implies $C = \frac{1}{n+1}$. Hence

$$\frac{(x+1)^{n+1}}{n+1} = \sum_{j=0}^{n}\binom{n}{j}\frac{x^{j+1}}{j+1}.$$

The result follows by setting $x=1$.